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Chaos, order statistics and unstable periodic orbits

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Abstract. We present an interesting connection between order statistics and unstable periodic orbits of chaotic maps. This can be used to locate all the unstable periodic points (of a given order) of one-dimensional chaotic maps with continuous invariant density. The densities of the ordered variates of the iterates are discontinuous exactly at unstable periodic points of the map. This is illustrated using the logistic map, where densities corresponding to a small number of iterates have been obtained in closed form. This scheme can also be applied to a class of continuous-time systems where the successive maxima of the time series behave as if they were generated from a unimodal map. We demonstrate this by using the Lorenz model.

1. Introduction

Existence of a dense set of unstable periodic orbits is one of the characteristic properties of a chaotic system [1]. These orbits represent the skeleton for the strange attractor of dissipative dynamical systems. Many quantities that characterize chaos in the system, such as the fractal dimension, the average Lyapunov exponent, the entropy and the invariant measure of the corresponding attractor can be determined by knowing the properties of unstable periodic orbits [2–6]. Further, the topological analysis of time series involves the computation of unstable periodic orbits; for a recent review see [7]. Extraction of unstable periodic orbits is a necessary step in several studies. For example, knowledge of the locations of various cycles is necessary for control of chaos [8]. Cycles are found to be useful in the synchronization of chaotic signals [9]. Moreover, quantization of classically chaotic conservative systems is accomplished, in the semiclassical regime, by a series expansion with respect to the lengths and stability coefficients of the periodic orbits [10]. The importance of unstable periodic orbits and their detection has attracted the attention of several researchers and a number of numerical methods have since been developed to extract unstable periodic orbits [2, 11–16]. In this paper we present an interesting connection between unstable periodic orbits and order statistics and demonstrate that it can be used to extract all the unstable periodic points of one-dimensional chaotic maps using order statistics.

The theory of extreme values and its generalization to order statistics is a classic subject and is extensively used in the study of independent and identically distributed random variables [17]. Whenever the points of sample space can be compared based on their numerical value, say as happens in the real line, one can think of new random variates such as maximum, second maximum etc. For example, let the outcomes of two dice, when thrown simultaneously, be D_1 and D_2. Then the random variate defined as $v = \max\{D_1, D_2\}$ is an ordered random variate. Let $\{X_0, X_1, \ldots, X_{n-1}\}$ be a sequence of n random numbers and their cumulative distribution function be $F(x)$. If the members of the above sequence are independent and identically distributed random variables, then the cumulative distribution of the ordered variate defined as the largest value in sequence is $[F(x)]^n$. Let X_n^k $(1 \leqslant k \leqslant n)$ be the kth largest value in the set i.e., $X_n^n \leqslant X_n^{n-1} \ldots \leqslant X_n^k \leqslant \ldots X_n^2 \leqslant X_n^1$. Order statistics is the study of the distributional properties of X_n^k.

Statistics of ordered variates such as extreme events/values are important in many areas of physical and applied sciences. For example, the breaking strength of a specimen is determined by its weakest element. Flood is the maximum discharge of water from a river. Extreme yields may characterize the occurence of bankruptcy or foreign-exchange realignments. More recently, it has been applied to diffusion process and economic modelling [18, 19].

The detection of unstable periodic orbits using order statistics works for all one-dimensional chaotic maps for which the existence of smooth invariant density is guaranteed. This includes measure (Lebesgue measure) preserving transformations like tent map (which is a discrete time one-dimensional analogue of Hamiltonian systems). Since there is no natural ordering of points in spaces of dimension greater than one, order statistics of the iterates of higher-dimensional systems is nontrivial and hence we will not address that issue here; see [20] for details. Extreme-value statistics is the study of the distributional properties of the maximum of a sequence. The formulation of the extreme-value statistics for one-dimensional chaotic systems is discussed in [21]. It has been shown that the extreme-value density is discontinuous on a set of points belonging to the unstable periodic orbits of the map [21]. However, we find that the extreme-value analysis does not locate all the unstable periodic points. The extreme-value density picks up only one point from each orbit corresponding to the maximum of the points in that orbit. Even in the case where we know the map exactly, this information cannot be used to generate other points of the same periodic orbit, as these orbits are unstable. Thus, one would like to have a scheme which would directly yield all the points of the orbit. Further, we find that the densities of other ordered variates also exhibit discontinuities and these points of discontinuity also belong to the unstable periodic points. This has motivated us to extend the analysis to order statistics instead of the simple extreme-value analysis in order to locate all the unstable periodic orbits.

2. One-dimensional chaotic maps

Let $f(x) : [a, b] \to [a, b]$ be a continuous, one-dimensional chaotic map with an invariant density $\rho_s(x)$. Let $\{x_0, x_1 = f(x_0), \ldots, x_{n-1} = f^{n-1}(x_0)\}$ be an n-point set. Let X_n^k be the kth largest member of the this set. Let the density of the kth largest member be denoted by $\rho_n^k(x)$. The order statistics of one-dimensional chaotic maps is formulated based on the elementary probability notions and is presented in appendix A.

The order density $\rho_n^k(x)$ for any n with $k \leqslant n$ is, in general, discontinuous on a set of points. The connection between order statistics and unstable periodic points can be stated as follows. Let S_n^k be the set of locations of the discontinuities of ρ_n^k. We observe that the set

$$O_n = \bigcup_{k=1}^{n} S_n^k \qquad \text{for all} \quad n \tag{1}$$

is equal to the set of all interior (excluding the end points a, b) periodic points of orders strictly less than n.

We now illustrate this interesting connection by applying it to the logistic map,

$$x_{n+1} = f(x_n) : [0, 1] \to [0, 1] = \lambda x_n (1 - x_n) \tag{2}$$

for $\lambda = 4$, a well-studied unimodal map exhibiting chaos [1] with a smooth invariant density

$$\rho_s(x) = \frac{1}{\pi \sqrt{x(1-x)}}. \tag{3}$$

The order densities for a small number of iterates have been obtained analytically based on the formalism discussed in appendix A. The $\rho_3^1(x)$, $\rho_3^2(x)$ and $\rho_3^3(x)$ of the logistic map are

$$\rho_3^1(x) = \begin{cases} \dfrac{1}{4\pi\sqrt{x(1-x)}} & 0 \leqslant x < \dfrac{3}{4} \\ \dfrac{7}{4\pi\sqrt{x(1-x)}} & \dfrac{3}{4} \leqslant x < \dfrac{5+\sqrt{5}}{8} \\ \dfrac{3}{\pi\sqrt{x(1-x)}} & \dfrac{5+\sqrt{5}}{8} \leqslant x < 1 \end{cases} \tag{4}$$

$$\rho_3^2(x) = \begin{cases} \dfrac{3}{4\pi\sqrt{x(1-x)}} & 0 \leqslant x < \dfrac{5-\sqrt{5}}{8} \\ \dfrac{2}{\pi\sqrt{x(1-x)}} & \dfrac{5-\sqrt{5}}{8} \leqslant x < \dfrac{3}{4} \\ \dfrac{5}{4\pi\sqrt{x(1-x)}} & \dfrac{3}{4} \leqslant x < \dfrac{5+\sqrt{5}}{8} \\ 0 & \dfrac{5+\sqrt{5}}{8} \leqslant x < 1 \end{cases} \tag{5}$$

$$\rho_3^3(x) = \begin{cases} \dfrac{2}{\pi\sqrt{x(1-x)}} & 0 \leqslant x < \dfrac{5-\sqrt{5}}{8} \\ \dfrac{3}{4\pi\sqrt{x(1-x)}} & \dfrac{5-\sqrt{5}}{8} \leqslant x < \dfrac{3}{4} \\ 0 & \dfrac{3}{4} \leqslant x < 1. \end{cases} \tag{6}$$

If the analysis is restricted to extreme values alone, the period-two point $(5-\sqrt{5})/8$ could not be located at all, see equation (4). In the present analysis, all the unstable periodic orbits of orders less than three, namely $\{(5-\sqrt{5})/8, \frac{3}{4}, (5+\sqrt{(5)})/8\}$ are picked up as discontinuities of $\rho_3^k(x)$ for some $k \leqslant 3$, as can be seen in equations (4)–(6).

In order to detect the locations of higher-order periodic orbits, we have to consider the sets of large number of iterates. The order densities of larger sequences can be obtained numerically. The numerical procedure for computing the order densities of one-dimensional chaotic maps is as follows. Given a map, the invariant density is obtained using the histogram of a large number of iterates after rejecting the transients. This exercise precedes the computation of order densities as the existence of invariant density is necessary and sufficient to render meaning to the averaging procedure. Then the order densities are computed as described below. Starting with an initial condition, $n-1$ successive iterates of the map are obtained and arranged in descending order. The kth largest member of the set is thus picked up. This is repeated for large number of realizations where the initial condition for the second realization is taken as the $(n-1)$th iterate. The whole procedure is repeated for many initial conditions. A histogram of the values of kth largest member of n iterates is thus obtained representing ρ_n^k. The typical number of realizations and initial conditions used in our numerical studies are 10^4 and 10^3, respectively.

The numerically computed order densities ρ_4^k for all $k \leqslant 4$ and the remarkable correspondence between the order densities and the unstable periodic points of the order less than four is shown in figure 1. This enables one to extract the unstable periodic points of any order by choosing an appropriate n. The locations of the cycles are, thus, read off from the extremely sharp discontinuities of the densities. This connection can be rigorously established and the relevant mathematical treatment is discussed in appendix B.

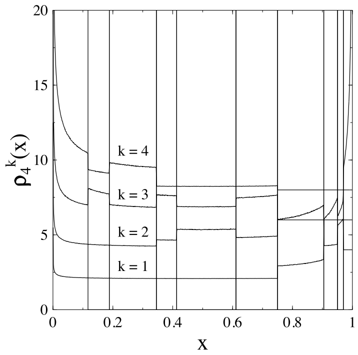

Figure 1. Numerically computed ρ_4^k for $k = 1, 2, 3$ and 4 of the logistic map (10^7 initial conditions, bin width = 0.001). The locations of the discontinuities and the cycle points of all orders <4 are compared. The cycle points are indicated by vertical lines. Note the absence of discontinuity at the origin which is a fixed point (see the text).

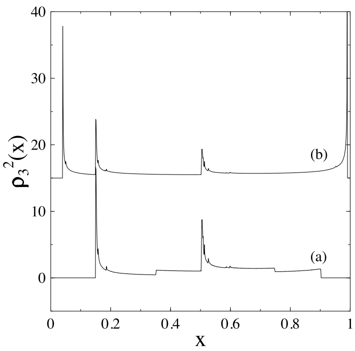

Figure 2. Order density of the logistic map for $\lambda = 3.96$. The Lyapunov exponent for $\lambda = 3.96$ is 0.533 75 and hence is in the chaotic regime. (*a*) $\rho_3^2(x)$; (*b*) corresponding invariant density.

Further, the topological entropy, which is a quantitative measure of chaos in the system, can be estimated† from the number of periodic orbits as follows:

$$h = \lim_{n\to\infty} \frac{1}{n} \ln N(n) \tag{7}$$

where $N(n)$ is the total number of periodic points of order n. We denote the number of elements in a set A by Card (A). Since $N(n)$ = Card (O_{n+1}) − Card (O_n), it can be obtained from the discontinuities of $\rho_{n+1}^k(x)(k = 1 \ldots n+1)$ and of $\rho_n^k(x)(k = 1 \ldots n)$. In the case of the logistic map we obtain Card (O_n) as 3, 9, 21, 51 and 105 for $n = 3$, 4, 5, 6 and 7, respectively. Since this formalism is based on the discontinuities of the density over an interval $[a, b]$, it cannot indicate a periodic point occurring at the extreme points a and b of the interval. Thus, in the above list (see also figure 1), the fixed point at the origin is not included. The topological entropy calculated with $n = 7$ is 0.6648, which is within 4% of the exact value ln 2. However, the value of the topological entropy approaches the exact value for larger and larger n.

We now turn to cases with continuous map with discontinuous invariant densities. For the logistic map with $\lambda < 4$, the invariant density, $\rho_s(x)$ is not available in closed form and one resorts to numerical schemes, see, for example, [23]. However, for our purpose, the existence of $\rho_s(x)$ can be numerically checked and is sufficient for computing order densities. For most $\lambda < 4$, $\rho_s(x)$ itself is discontinuous on a large number of points. Thus, the corresponding order densities pick up discontinuities at those points at which $\rho_s(x)$ is discontinuous and also at unstable periodic points. Since the strength of the inherent discontinuities of $\rho_s(x)$ in $\rho_n^k(x)$ is large compared with that of the discontinuities at periodic points, it is numerically difficult to count or detect their locations. The $\rho_3^2(x)$ of the logistic map for $\lambda = 3.96$ and the corresponding invariant density are shown in figure 2. It can be seen that the discontinuties in $\rho_3^2(x)$ corresponding to the unstable periodic points are weak.

The analysis so far is confined to the map which is continuous with smooth invariant density. If the map is discontinuous, then the order densities of its iterates pick up additional discontinuities which do not belong to the unstable periodic orbits. The additional discontinuous points of $\rho_n^k(x)$ belong to the points at which either the map, f, or any of its

† To the best of our knowledge, this estimate is an upper bound on the topological entropy, see [22] for details.

Table 1. Periodic orbits and discontinuities of shift map.

Map	Fixed points	Discontinuities of map
f	—	$\frac{1}{2}$
f^2	$\frac{1}{3}$ and $\frac{2}{3}$	$\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$
f^3	$\frac{4}{7}$ and $\frac{6}{7}$	$\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{5}{8}$, $\frac{3}{4}$ and $\frac{7}{8}$

$n-1$ composite maps is discontinuous. This is illustrated below by using the Bernoulli shift map which is defined as

$$x_{n+1} = f(x_n) = 2x_n \bmod 1 \qquad x \in [0, 1). \tag{8}$$

For example, $\rho_3^k(x)$, $k \leqslant 3$ exhibit discontinuities at the fixed points and the discontinuities of the maps f^3, f^2 and f, see table 1. Thus, when the map is discontinuous (in the sense of a function having discontinuity), with a smooth invariant density, the order densities show discontinuities corresponding to points of discontinuity of the map as well as unstable periodic points. On the other hand, if the map is continuous with an invariant density which is discontinuous at some points, the corresponding order densities exhibit discontinuities at those points plus the unstable periodic points. In summary, the discontinuities of the invariant density and that of the map appear as discontinuities in the order densities in addition to the unstable periodic points.

3. Lorenz system

There are situations where the essence of the dynamics of a continuous-time dynamical system is captured effectively by one-dimensional maps or their equivalent. In his classic paper Lorenz showed that the successive peaks of a one-dimensional time series behave like iterates of a map [24], see also [25]. We work with the Lorenz model [24]

$$\dot{x} = \sigma(y - x) \qquad \dot{y} = x(r - z) - y \qquad \dot{z} = xy - bz \tag{9}$$

for parameters $r = 28$, $\sigma = 10$ and $b = \frac{8}{3}$.

We compute the $\rho_n^k(x)$ for different n for the map constructed out of the successive maxima of the time series corresponding to the state space variable z. Existence of an invariant density for the so constructed map is verified numerically. $\rho_n^k(x)$ of this map is also discontinuous on a set of points. $\rho_5^k(x)$ for all $k \leqslant 5$ is shown in figure 3. The topological entropy of the map is calculated using equation (7). For $n = 4, 5$ and 6, the topological entropy $h = 0.6212, 0.6664$ and 0.6679 respectively, showing a reasonable convergence in the numerical value of topological entropy.

It is important to consider the effect of noise on the order densities to explore the possible applicability of this method to an experimental time series convoluted with noise. Here, the distribution and strength of the noise play a decisive role. Our preliminary investigations involve the addition of noise, generated from uniform distribution, to the iterates of the logistic map with $\lambda = 4$. The spillover from $[0, 1]$ is reinjected. The discontinuity at the periodic points is observed to smoothen with an increase in the strength of the noise. The effect of noise on $\rho_3^2(x)$ of the logistic map with $\lambda = 4$ is shown in figure 4.

In summary, we have presented a connection between order statistics and the unstable periodic orbits of a chaotic map which can be used to extract all the unstable periodic points of chaotic maps with smooth invariant density. This method uses order statistics. We calculate ρ_n^k of the logistic map analytically for $n = 3$ and numerically for $n > 3$ and illustrate that their discontinuities coincide with the unstable periodic points of all orders less than n. Further,

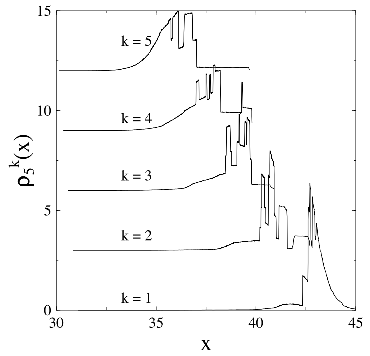

Figure 3. $\rho_5^k(x)$ for all $k \leqslant 5$ of the map constructed using the successive maxima of the time series $z(t)$ of the Lorenz system.

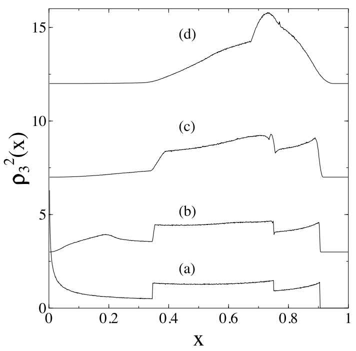

Figure 4. $\rho_3^2(x)$ of the logistic map with $\lambda = 4$ for different strengths of additive noise generated from uniform distribution. The random number added is $r = \alpha x$, where α is proportional to the strength of the noise and $x \in [0, 1]$ is a uniform random number. (*a*) 0% noise; (*b*) 0.2% noise; (*c*) 1% noise; (*d*) 5% noise.

we demonstrate the applicability of this method to the map constructed out of the successive maxima of the time series $z(t)$ of Lorenz equations. The unstable periodic orbits enumerated are used to estimate topological entropy. We present the formulation of the order statistics as well as an outline of the proof which establishes the connection between order statistics and unstable periodic orbits for any continuous map with a smooth invariant density in appendices A and B, respectively.

Acknowledgments

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Appendix A. Formulation of order statistics

Let $f(x) : [a, b] \to [a, b]$ be a continuous, one-dimensional chaotic map with an invariant density $\rho_s(x)$. Let $\{x_0, x_1 = f(x_0), \ldots, x_{n-1} = f^{n-1}(x_0)\}$ be an n-point set. Let X_n^k be the kth largest member of the this set. Let the density of the kth largest member be denoted by $\rho_n^k(x)$.

The formulation of the order statistics involves $P_n^k([x, x + \mathrm{d}x])$, denoting the probability that the value of the kth largest member of n iterates lies in the interval $[x, x + \mathrm{d}x]$.

$$P_n^k([x, x + \mathrm{d}x]) = \frac{1}{(k-1)!} \sum_{i_1=0}^{n-1} \sum_{\mathcal{I}} \mathrm{Prob}[x_{i_1} \in (x, x + \mathrm{d}x); x_i > x | i \in \mathcal{I}; x_j < x | j \in \mathcal{J}] \tag{10}$$

where $\mathcal{I} = \{i_2, \ldots, i_k\}$ and $\mathcal{J} = \{i_{k+1}, \ldots, i_n\}$ with $\{i_1, \ldots, i_n\}$ being a permutation of $\{0, \ldots, n-1\}$. The summation over $\mathcal{I}$ requires some amplification. When the density of the kth largest member of the set is considered, there will be $k - 1$ members greater than X_n^k. All possible permutations of $\{0, \ldots, n-1\}$ is equivalent to sum over $\mathcal{I}$ with every $i_k \in \mathcal{I}$ running from 0 to $n - 1$. This summation, however, has contribution from the permutations of the $k-1$ members of $\mathcal{I}$ among themselves which are $(k-1)!$ in number and hence we divide the expression with $(k-1)!$, see equation (10).

The cumulative distribution $F_n^k(x) = \mathrm{Prob}(X_n^k \leqslant x)$ and is given by

$$F_n^k(x) = \frac{1}{(k-1)!}\sum_{i_1=0}^{n-1}\sum_{\mathcal{I}}\int_a^x \mathrm{d}x' \left[\left(\prod_{i\in\mathcal{I}}\int_{x'}^b \mathrm{d}x_i\right)\left(\prod_{j\in\mathcal{J}}\int_a^{x'} \mathrm{d}x_j\right)\rho(x_1,\ldots,x_n)|_{x_{i_1}=x'}\right] \quad (11)$$

where $\rho(x_1, x_2, \ldots, x_n)$, $\{x_i\} \in (a, b)$, is an n-point joint density and can be written for a deterministic map f with an invariant density $\rho_s(x)$ as

$$\rho(x_0, x_2, \ldots, x_{n-1}) = \rho_s(x_0)\prod_{j=1}^{n-1}\delta(x_j - f^{(j)}(x_0)) \quad (12)$$

with $\delta(\cdot)$ being the Dirac delta function.

Thus, $\rho_n^k(x)$ is given by the derivative of $F_n^k(x)$.

$$\rho_n^k(x) = \frac{1}{(k-1)!}\sum_{\mathcal{I}}\rho_s(x)\prod_{i\in\mathcal{I}}\Theta(f^i(x)-x)\prod_{j\in\mathcal{J}}\Theta(x-f^j(x)) + \frac{1}{(k-1)!}\sum_{i_1=1}^{n-1}\sum_{\mathcal{I}}\sum_{\alpha=1}^{I_{i_1}} \times \frac{\rho_s(g_{i_1\alpha}(x))}{|\frac{\mathrm{d}}{\mathrm{d}z}f^{i_1}(z)|_{z=g_{i_1\alpha}(x)}}\prod_{i\in\mathcal{I}}\Theta(f^i(g_{i_1\alpha}(x))-x)\prod_{j\in\mathcal{J}}\Theta(x-f^j(g_{i_1\alpha}(x))) \quad (13)$$

where $\Theta(\cdot)$ is the Heaviside step function. The set $\{g_{i_1\alpha}(x)/f^{i_1}(g_{i_1\alpha}(x)) = x\}$ are the preimages of x with respect to f^{i_1} and I_{i_1} is the number of such preimages.

Appendix B. Connection between order statistics and unstable periodic orbits

Our central result is: given a continuous one-dimensional map $f : [a, b] \to [a, b]$ with a continuous invariant density $\rho_s(x)$, the order density $\rho_n^k(x)$ is discontinuous at an interior point $x_\star$ if and only if $x_\star$ is an unstable periodic point such that $f^l(x_\star) = x_\star$ with $l \leqslant n-1$.

We now give an outline of the proof where we first explain the notation used. (i) A periodic point of order l is denoted by $p_{jl\beta}$, i.e., $f^l(p_{jl\beta}) = p_{jl\beta}$. The index $\beta = 1, \ldots, l$ corresponds to the distinct points of the l-cycle. In a chaotic map, in general, there exist more than one l-cycle, and the number of l-cycles N_l increases with l. The index j runs over the different l-cycles and hence $j = 1, \ldots, N_l$. For example, in the logistic map, we have two orbits of period three. The periodic points of a given l-cycle are ordered such that $p_{jll} = \max_\beta\{p_{jl\beta}\}$ corresponds to the maximum of that cycle. (ii) Let $y = f^i(x)$. The set of preimages of x with respect to f^i are $\{g_{i\alpha}(x)|f^i(g_{i\alpha}(x)) = x\}$, $\alpha = 1, \ldots, I_i$ where I_i is the number of preimages. For example, the preimages of $x = \frac{3}{4}$ of the logistic map are $\{\frac{3}{4}, \frac{1}{4}\}$.

As a first step, we show that the extreme-value density, $\rho_n^1(x)$ is discontinuous at x if and only if x is the maximum point of a periodic orbit, i.e., $x = p_{jll}$, $l \leqslant n-1$, $j = 1, \ldots N_l$. Proving this involves two stages namely, to show that the extreme-value density cannot be discontinuous at x unless x is the maximum of a periodic orbit and then show that $\rho_n^1(x)$ is discontinuous at the maximum of all the periodic points. The extreme-value density can be formulated as

$$\rho_n^1(x) = \rho_s(x)\prod_{j=1}^{n-1}\Theta(x-f^j(x)) + \sum_{i=1}^{n-1}\sum_{\alpha=1}^{I_i}A_{i\alpha}(x)\Theta(x-g_{i\alpha}(x))\prod_{j\neq i}\Theta(x-f^j(g_{i\alpha}(x))) \quad (14)$$

where

$$A_{i\alpha}(x) = \rho_s(g_{i\alpha}(x)) \Big/ \left|\frac{\mathrm{d}}{\mathrm{d}z}f^i(z)\right|_{z=g_{i\alpha}(x)}. \quad (15)$$

If the invariant density is continuous, then so are $A_{i\alpha}(x)$. Equation (14) can be recast as

$$\rho_n^1(x) = \rho_s(x)C_0(x) + \sum_{i=1}^{n-1}\sum_{\alpha=1}^{I_i} A_{i\alpha}(x)C_{i\alpha}(x). \tag{16}$$

As all the terms in equation (14) are positive definite, proving either $C_0(x)$ or $C_{i\alpha}(x)$ are discontinuous at a point is sufficient to prove $\rho_n^1(x)$ is discontinuous. Also note $\Theta(z) = (1 + \mathrm{sgn}\,(z))/2$ is discontinuous only at $z = 0$.

Since $\rho_s(x)$ and hence $A_{i\alpha}(x)$ are continuous, $\rho_n^1(x)$ is discontinuous only if either $\Theta(x - f^l(x))$ for some $l \leqslant n-1$, or $\Theta(x - g_{i\alpha}(x))$ for some $i \leqslant n-1$, or $\Theta(x - f^m(g_{i\alpha}(x)))$ for some $m \neq i, m \leqslant n-1$ is discontinuous.

If $\Theta(x - f^l(x))$ is discontinuous then $x = f^l(x)$ implying $x \in \{p_{jl\beta}\}$. Considering $C_0(p_{jl\beta} + \epsilon)$ (where ϵ is an infinitesimal quantity of appropriate sign), we have

$$C_0(p_{jl\beta} + \epsilon) = \prod_{k=1}^{n-1} \Theta(p_{jl\beta} - f^k(p_{jl\beta}) + \mathrm{O}(\epsilon)). \tag{17}$$

If $p_{jl\beta} \neq p_{jll}$, in the product, there exists a k_l such that $f^{k_l}(p_{jl\beta}) = p_{jll}$ and the product vanishes as $p_{jll} > p_{jl\beta}$ for all $\beta < l$. This shows that $C_0(x)$ cannot be discontinuous unless $x = p_{jll}$.

$\Theta(x - g_{i\alpha}(x))$ is discontinuous at $x = g_{i\alpha}(x)$. We have $f^i(x) = f^i(g_{i\alpha}(x)) = x$ and this implies $x \in \{p_{ji\beta}\}$. Considering $C_{i\alpha}(p_{ji\beta} + \epsilon)$, the leading-order term is

$$C_{i\alpha}(p_{ji\beta} + \epsilon) = \Theta\left(\epsilon\left[1 - \frac{\mathrm{d}}{\mathrm{d}x} g_{i\alpha}(x)|_{x=p_{ji\beta}}\right]\right) \prod_{k\neq i} \Theta(p_{ji\beta} - f^k(p_{ji\beta}) + \mathrm{O}(\epsilon)). \tag{18}$$

It can be shown that $g'_{i\alpha}(x) = 1/f'(g_{i\alpha}(x))$ and $|f'(g_{i\alpha}(p_{ji\beta}))|$ is always greater than one for all the unstable periodic orbits. This means the first Θ function in the above equation is always nonzero. Thus, by a similar reasoning as outlined above, the second term in the product cannot survive unless $x = p_{jii}$.

Consider $\Theta(x - f^m(g_{i\alpha}(x)))$ which is discontinuous at $x = f^m(g_{i\alpha}(x))$ for some $m \in \{1, \ldots, n-1\}\backslash\{i\}$. There exists an l such that $x = f^m(g_{i\alpha}(x))$ implies $x = f^l(x)$, where $l = \max\{m - i, i - m\}$ implying $x \in \{p_{jl\beta}\}$. By using the previous arguments, one can show that the discontinuity cannot occur unless $x = p_{jll}$. This proves that $\rho_n^1(x)$ cannot be discontinuous unless $x = p_{jll}$ for some l and j.

Conversely, we show that at every $x = p_{jll}$, $\rho_n^1(x)$ is discontinuous. This is done in two steps for convenience: (a) if p_{jll} is such that $l > (n-1)/2$, $C_0(x)$ registers a discontinuity at $x = p_{jll}$ while (b) if $l \leqslant (n-1)/2$, $C_{i\alpha}(x)$ will be discontinuous at $x = p_{jll}$.

(a) Consider $C_0(p_{jll} + \epsilon)$ and writing it as a product of three terms, we have

$$C_0(p_{jll} + \epsilon) = \Theta\left(\epsilon\left[1 - \frac{\mathrm{d}}{\mathrm{d}x} f^l(x)|_{x=p_{jll}}\right]\right) \prod_{k=1}^{l-1} \Theta(p_{jll} - f^k(p_{jll}) + \mathrm{O}(\epsilon))$$
$$\times \prod_{k=1}^{n-l-1} \Theta(p_{jll} - f^{k+l}(p_{jll}) + \mathrm{O}(\epsilon)). \tag{19}$$

$f^k(p_{jll}) < p_{jll}$ for all $k \leqslant l-1$ and $n-l-1 < l-1$ if $l > (n-1)/2$. This implies that the product terms in the equation (19) are nonzero and the first term can also be made nonzero by choosing an ϵ of appropriate sign. This proves that if $l > (n-1)/2$, at $x = p_{jll}, l \leqslant n-1, j = 1, \ldots, N_l$, $C_0(x)$ and hence $\rho_n^1(x)$ are discontinuous.

(b) We prove that, given $x = p_{jmm}, m \neq i$ and $m \leqslant (n-1)/2$, there exists a Θ function in $C_{i\alpha}(x)$, see equation (14), with argument $x - f^l(g_{i\alpha}(x)), l \neq i$ such that $l = i \pm m$ which is discontinuous at $x = p_{jmm}$ implying $C_{i\alpha}(x)$ and hence $\rho_n^1(x)$ to be discontinuous at $x = p_{jmm}$.

Similarly, one can prove that the density corresponding to $k = n$, the minimum of the iterates, $\rho_n^n(x)$ is discontinuous at x if and only if $x = p_{jl1}$, i.e., at the minimum of the periodic orbits.

Finally, it can be proven that at every periodic point $x = p_{jl\beta}$, there exists a k such that the order density $\rho_n^k(x)$ is discontinuous at $p_{jl\beta}$.

$\rho_n^k(x)$, the order density of the kth maximum of an n-point sequence of iterates is given by equation (13). Given a periodic point p, one can show that the index set $\{0, \ldots, n-1\}$ can be written in the following way:

$$\begin{aligned} I' &= \{i \mid f^i(p) > p\} \\ J' &= \{j \mid f^j(p) < p\} \\ L' &= \{l \mid f^l(p) = p\} \end{aligned} \tag{20}$$

such that

$$\{0, \ldots, n-1\} = I' \cup J' \cup L'. \tag{21}$$

This partition is equivalent to chosing a k for a given p and n. Once n and k are fixed, the order density $\rho_n^k(x)$ can be shown to be discontinuous at $x = p$ by an argument similar to that outlined in the case of ρ_n^1.

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